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TWO TYPES OF PRESERVING OPERATORS AND THEIR STABILITY*

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Abstract

In this paper, we first introduce orthogonality preserving operators, approximately orthogonality preserving operators on inner product spaces and their properties. Next, we discuss some properties of orthogonality preserving operators and approximately orthogonality preserving operators that are true for right-angle preserving when the operator is linear. Finally, we show the stability between approximate right-angle preserving and right-angle. preserving.

Keywords: Orthogonality preserving operators; Right-angle preserving operators; Stability

1 Preliminaries.

In this section, we introduce some definitions and examples. Let X and Y be two inner product spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the inner products denoted by $\langle \cdot,\cdot \rangle$ and the associated norms denoted by $\| \cdot \|$. As usual, vectors x and y are said to be ***orthogonal*** $(x \perp y)$, if $\langle x, y\rangle = 0$. $\mathbb{K}$ denotes the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers. For some $\varepsilon \in [0,1)$, an element x of an inner product space X is said to be ***approximate orthogonal*** to an element $y \in X$ if $|\langle x, y\rangle| \le \varepsilon \|x\| \; \|y\|$. We also say that x and y are **ε-*orthogonal*** and we write $x \perp^{\varepsilon} y$. Let H be an infinite dimensional Hilbert space and $B(H)$ be the space of all bounded linear operators acting on H. T is a ***similarity*** if there exists $\gamma \ge 0$ such that $\|T(x)\| = \gamma \|x\|$, for all $x \in X$. We choose $\gamma = 1$, we get $\|T(x)\| = \|x\|$, for all $x \in X, T$ is ***isometry***. We have known that every unitary operator on H is an isometry. If $T \in B(H)$, then $\langle T(x), y\rangle$ is linear in x, conjugate-linear in y, and bounded. There exists a unique $T^* \in B(H)$ for which $\langle T(x), y\rangle = \langle x, T^*(y)\rangle, x, y \in H$ and so that $\|T^*\| = \|T\|$.

1.1 Definition. $[2]$ *Let X and Y be two inner product spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A mapping $T : X \to Y$ is called* **orthogonality preserving (OP)** *if $x \perp y$ implies $T(x) \perp T(y)$, for all $x, y \in X$ and is called* **strongly orthogonality preserving (SOP)** *if $x \perp y \Leftrightarrow T(x) \perp T(y)$, for all $x, y \in X$.*

1.2 Definition. $[3]$ *Let X and Y be real inner product spaces and $T : X \to Y$ be an operator. T is said to be the* ***right-angle preserving operator (RAP)*** *if for all $x, y, z \in X$ such that $x - z \perp y - z$ implies $T(x) - T(z) \perp T(y) - T(z)$.*

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1.3 Definition. $[2]$ *Let* X *and* Y *be two inner product spaces over the same field* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *A mapping* $T : X \to Y$ *is called approximately orthogonality preserving (ε-OP) if and only if for all* $x, y \in X$ *such that* $x \perp y \Rightarrow T(x) \perp^{\varepsilon} T(y)$ *with some* $\varepsilon \in [0,1)$.

The following example shows that a linear solution of approximately orthogonality preserving need not satisfy orthogonality preserving.

1.4 Example. *Let* $\varepsilon \in (0,1)$ *and* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be given by*

$$T(x_1, x_2) = \left(x_1, \sqrt{1-\varepsilon}\, x_2\right).$$

Fix $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ *such that* $x \perp y$. *That is,* $x_1 y_1 = -x_2 y_2$.

If $x_1 x_2 y_1 y_2 = 0$, *then* $x_1 y_1 = 0$ *and hence*

$$\begin{aligned} \langle T(x), T(y) \rangle &= \left\langle (x_1, \sqrt{1-\varepsilon}\, x_2), (y_1, \sqrt{1-\varepsilon}\, y_2) \right\rangle \\ &= x_1 y_1 + (1-\varepsilon) x_2 y_2 \\ &= x_1 y_1 + (1-\varepsilon)(-x_1 y_1) \\ &= 0. \end{aligned}$$

Therefore $T(x) \perp T(y)$.

Thus we assume $x_1 x_2 y_1 y_2 \neq 0$ *and define* $a = \dfrac{x_1}{x_2} = \dfrac{-y_2}{y_1} \neq 0.$

We have

$$\begin{aligned} \frac{\left|\langle T(x), T(y) \rangle\right|}{\|T(x)\| \; \|T(y)\|} &= \frac{\left|x_1 y_1 + (1-\varepsilon) x_2 y_2\right|}{\sqrt{x_1^2 + (1-\varepsilon) x_2^2}\sqrt{y_1^2 + (1-\varepsilon) y_2^2}} \\ &= \frac{\left|-x_2 y_2 + (1-\varepsilon) x_2 y_2\right|}{\sqrt{x_1^2 + (1-\varepsilon) x_2^2}\sqrt{y_1^2 + (1-\varepsilon) y_2^2}} \\ &= \frac{\varepsilon \left|x_2 y_2\right|}{\sqrt{x_1^2 + (1-\varepsilon) x_2^2}\sqrt{y_1^2 + (1-\varepsilon) y_2^2}} \\ &= \frac{\varepsilon}{\sqrt{\dfrac{x_1^2 + (1-\varepsilon) x_2^2}{x_2^2}}\sqrt{\dfrac{y_1^2 + (1-\varepsilon) y_2^2}{y_2^2}}} \\ &= \frac{\varepsilon}{\sqrt{a^2 + (1-\varepsilon)}\sqrt{\dfrac{1}{a^2} + (1-\varepsilon)}} \\ &= \frac{\varepsilon}{\sqrt{1 + (1-\varepsilon)^2 + (1-\varepsilon)(a^2 + \dfrac{1}{a^2})}}. \end{aligned}$$

Since $(1-\varepsilon)>0,\ (a^2+\frac{1}{a^2})>0$ *and* $\sqrt{1+(1-\varepsilon)^2+(1-\varepsilon)(a^2+\frac{1}{a^2})}>1,\ \frac{|\langle T(x),T(y)\rangle|}{\|T(x)\|\ \|T(y)\|}<\varepsilon.$

Therefore $T(x)\perp^{\varepsilon} T(y).$ *Clearly, T does not preserve inner product.*

1.5 Definition. $[3]$ *Let* X *and* Y *be real inner product spaces and* $T:X\to Y$ *be an operator. For* $\varepsilon\in[0,1),$ T *is said to be the* ***approximate right-angle preserving(ε-RAP)*** *if for all* $x,y,z\in X$ *such that*

$$x-z\perp y-z \ \textit{ implies } \ T(x)-T(z)\perp^{\varepsilon} T(y)-T(z).$$

1.6 Example. *Let* $\varepsilon\in[0,1)$, and $T:\mathbb{R}^2\to\mathbb{R}^2$ *be defined by*

$$T(x_1,x_2)=(4-\sqrt{\varepsilon}x_1,4-2\sqrt{\varepsilon}x_2).$$

Let $x=(x_1,x_2),y=(y_1,y_2),z=(z_1,z_2)\in\mathbb{R}^2$ *such that* $x-z\perp y-z.$ *That is*

$$(x_1-z_1)(y_1-z_1)=-(x_2-z_2)(y_2-z_2).$$

Now, we assume that $(x_1-z_1)(y_1-z_1)(x_2-z_2)(y_2-z_2)\neq 0$ *and define*

$$a=\frac{x_1-z_1}{x_2-z_2}=-\frac{(y_2-z_2)}{y_1-z_1}\neq 0.$$

Then we have

$$\frac{|\langle T(x)-T(z),T(y)-T(z)\rangle|}{\|T(x)-T(z)\|\ \|T(y)-T(z)\|}$$

$$=\frac{\left|\left\langle\left(-\sqrt{\varepsilon}(x_1-z_1),-2\sqrt{\varepsilon}\,(x_2-z_2)\right),\left(-\sqrt{\varepsilon}(y_1-z_1),-2\sqrt{\varepsilon}(y_2-z_2)\right)\right\rangle\right|}{\sqrt{\varepsilon(x_1-z_1)^2+4\varepsilon(x_2-z_2)^2}\sqrt{\varepsilon(y_1-z_1)^2+4\varepsilon(y_2-z_2)^2}}$$

$$=\frac{|\varepsilon(x_1-z_1)(y_1-z_1)+4\varepsilon(x_2-z_2)(y_2-z_2)|}{\left[\varepsilon^2(x_1-z_1)^2(y_1-z_1)^2+16\varepsilon^2(x_2-z_2)^2(y_2-z_2)^2+4\varepsilon^2(x_2-z_2)^2(y_1-z_1)^2+4\varepsilon^2(x_1-z_1)^2(y_2-z_2)^2\right]^{\frac{1}{2}}}$$

$$=\frac{|3\varepsilon(x_2-z_2)(y_2-z_2)|}{\left[17\varepsilon^2(x_2-z_2)^2(y_2-z_2)^2+4\varepsilon^2\left(\left(\frac{x_2-z_2}{x_1-z_1}\right)^2+\left(\frac{y_2-z_2}{y_1-z_1}\right)^2\right)(x_2-z_2)^2(y_2-z_2)^2\right]^{\frac{1}{2}}}$$

$$=\frac{|3\varepsilon(x_2-z_2)(y_2-z_2)|}{\left[17+4\left(\frac{1}{a^2}+a^2\right)\right]^{\frac{1}{2}}\varepsilon(x_2-z_2)(y_2-z_2)}=\frac{3}{\left[17+4\left(\frac{1}{a^2}+a^2\right)\right]^{\frac{1}{2}}}<\varepsilon.$$

Thus $T(x)-T(z)\perp^{\varepsilon} T(y)-T(z).$ *Hence T is approximate right-angle preserving.*

2 Orthogonality Preserving Operators and Right-Angle Preserving Operators

In this section, we discuss the properties of orthogonality and right-angle preserving operators. We investigate right-angle preserving normal operators in $B(H)$ as orthogonality preserving normal operators in $B(H)$.

2.1 Properties of Orthogonality Preserving Operators

We start with equivalent conditions for an orthogonality preserving operator. We present the relation between an orthogonality preserving operator and its adjoint, and the relation between a bounded linear orthogonality preserving normal operator and its adjoint.

2.1 Theorem. $[2]$ *Let X and Y be two inner product spaces over the same field $\mathbb{K}$.*

For a nonvanishing operator $T: X \to Y$ the following conditions are equivalent:

(1) *T is linear and orthogonality preserving.*

(2) *T is linear and there exists $\gamma > 0$ such that $\|T(x)\| = \gamma \|x\|$, for all $x \in X$.*

(3) *There exists $\gamma > 0$ such that $\langle T(x), T(y) \rangle = \gamma^2 \langle x, y \rangle$, for all $x, y \in X$.*

Proof. See $[2]$. ■

2.2 Theorem. $[1]$ *Let H be a Hilbert space and $T \in B(H)$. If T is an orthogonality preserving operator, then so are T^*T and $|T|$, where T^* is adjoint of T* and $|T| = \sqrt{T^*T}$.

Proof. Let x, y be two orthogonal elements of H. Since T is orthogonality preserving,

we have

$$\langle T(x), T(y) \rangle = 0 \text{ which implies } \langle x, T^*T(y) \rangle = 0.$$

By using orthogonality preserving of T,

$$\langle T(x), TT^*T(y) \rangle = 0 \text{ which implies } \langle T^*T(x), T^*T(y) \rangle = 0.$$

Hence T^*T is orthogonality preserving.

Again, for $|T| = \sqrt{T^*T}, \langle x, T^*T(y) \rangle = 0$ gives

$$\langle x, |T|^2 (y) \rangle = 0.$$

Therefore

$$\langle |T|(x), |T|(y) \rangle = 0.$$

Thus $|T|$ is orthogonality preserving. ■

2.3 Theorem. $[1]$ *Let H be a Hilbert space and T be a bounded linear orthogonality preserving operator on H. Then T^* is a bounded linear orthogonality preserving operator on H if and only if T* is normal.

Proof. Let T be orthogonality preserving. Suppose that T is normal.

That is

$$T^*T = TT^*.$$

We show that T^* is orthogonality preserving. Let x, y be two orthogonal elements of H. Since T is orthogonality preserving,

$$\langle T(x), T(y) \rangle = 0 \text{ which implies } \langle x, T^*T(y) \rangle = 0.$$

Since T is normal,

$$\langle x, TT^*(y) \rangle = 0 \text{ which implies } \langle T^*(x), T^*(y) \rangle = 0.$$

Hence T^* is orthogonality preserving.

Conversely, suppose that T and T^* are orthogonality preserving. By the condition (2) of Theorem 2.1, there exist $\gamma, \gamma' > 0$ such that

$$\|T(x)\| = \gamma \|x\|$$

and

$$\|T^*(x)\| = \gamma' \|x\|$$

for all $x \in H$. Thus

$$\|T\| = \gamma \text{ and } \|T^*\| = \gamma'.$$

But $\|T\| = \|T^*\|$, so that $\gamma' = \gamma$. By Theorem 2.2, $TT^*, T^*T, |T|$ and $|T^*|$ are orthogonality preserving. Since $T^*(x) \in H$,

$$\|TT^*(x)\| = \gamma \|T^*(x)\| = \gamma \cdot \gamma \|x\| = \gamma^2 \|x\|.$$

Also, for $|T| = \left(T^*T\right)^{\frac{1}{2}}$ and $|T^*| = \left(TT^*\right)^{\frac{1}{2}}$, the scalar is equal to γ.

Now, we have

$$\begin{aligned}\left\langle \left(T^*T - TT^*\right)(x), \left(T^*T - TT^*\right)(x) \right\rangle &= \|T^*T(x)\|^2 + \|TT^*(x)\|^2 - \langle T^*T(x), TT^*(x) \rangle - \langle TT^*(x), T^*T(x) \rangle \\ &= 2\gamma^4 \|x\|^2 - \langle T^*T(x), TT^*(x) \rangle - \langle TT^*(x), T^*T(x) \rangle.\end{aligned}$$

Also, by using polarization identity

$$
\begin{aligned}
\left\langle T^{*}T(x), TT^{*}(x)\right\rangle &= \left\langle |T|^{2}(x), TT^{*}(x)\right\rangle \\
&= \left\langle |T|(x), |T|TT^{*}(x)\right\rangle \\
&= \frac{1}{4}\sum_{k=0}^{3} i^{k}\left\langle |T|(x)+i^{k}|T|TT^{*}(x), |T|(x)+i^{k}|T|TT^{*}(x)\right\rangle \\
&= \frac{1}{4}\sum_{k=0}^{3} i^{k}\left\| |T|\left(x+i^{k}TT^{*}(x)\right)\right\|^{2} \\
&= \frac{1}{4}\gamma^{2}\sum_{k=0}^{3} i^{k}\left\langle x+i^{k}TT^{*}(x), x+i^{k}TT^{*}(x)\right\rangle \\
&= \gamma^{2}\left\langle x, TT^{*}(x)\right\rangle \\
&= \gamma^{4}\|x\|^{2}.
\end{aligned}
$$

Hence $\left\langle T^{*}T(x), TT^{*}(x)\right\rangle = \gamma^{4}\|x\|^{2}$. Also, $\left\langle TT^{*}(x), T^{*}T(x)\right\rangle = \gamma^{4}\|x\|^{2}$ by changing the role of T by T^{*}. Therefore, we have

$$\left\langle T^{*}T(x)-TT^{*}(x), T^{*}T(x)-TT^{*}(x)\right\rangle = 2\gamma^{4}\|x\|^{2}-\gamma^{4}\|x\|^{2}-\gamma^{4}\|x\|^{2}=0$$

for every $x \in H$. Thus $\left\|T^{*}T-TT^{*}\right\|=0$ and so $TT^{*}=T^{*}T.$ Hence T is normal. ■

2.2 Properties of Right-Angle Preserving Operators

In this subsection, we show every right-angle preserving operator is orthogonality preserving. We express equivalent conditions for a right-angle preserving operator. We examine the relation between a bounded linear right-angle preserving normal operator and its adjoint and the relation between a right-angle preserving operator and its adjoint. We prove that every unitary operator on H is an orthogonality preserving operator and this result is true for a right-angle preserving operator.

2.4 Lemma. *Let H be a Hilbert space and $T \in B(H)$. If T is right-angle preserving, then T is orthogonality preserving.*

Proof. Suppose that T is right-angle preserving and T is linear. Let $u = x - z$ and $v = y - z$ with $u \perp v.$ By right-angle preserving,

$$T(x)-T(z)\perp T(y)-T(z).$$

But T is linear, so that

$$T(x-z)\perp T(y-z).$$

That is, $T(u)\perp T(v).$ Thus T is orthogonality preserving. ■

2.5 Theorem. $[3]$ *Let X and Y be real inner product spaces and $T: X \to Y$ be an operator. The following conditions are equivalent*:

(1) *T satisfies right-angle preserving and $T(0)=0$*;

(2) *T satisfies orthogonality preserving and T is continuous and linear*;

(3) *T satisfies orthogonality preserving and T is linear*;

(4) *T is linear and satisfies $\|T(x)\| = \gamma\|x\|, x \in X$ for some constant $\gamma \geq 0$*;

(5) *T satisfies $\langle T(x), T(y)\rangle = \gamma^2 \langle x, y\rangle, x, y \in X$ for some constant $\gamma \geq 0$*;

(6) *T satisfies orthogonality preserving and T is additive.*

Proof. See [3]. ■

2.6 Theorem. *Let H be a Hilbert space and T be a bounded linear right-angle preserving operator on H. Then T^* is a bounded linear right-angle preserving operator on H if and only if T is normal.*

Proof. Let T be right-angle preserving. Suppose that T is normal. Then $T^*T = TT^*$. We show that T^* is right-angle preserving. Let $x - z$ and $y - z$ be two orthogonal elements of H. Since T is right-angle preserving, we have

$$\langle T(x) - T(z), T(y) - T(z)\rangle = 0.$$

Since T is linear, $\langle T(x-z), T(y-z)\rangle = 0$. It follows that $\langle x - z, T^*T(y-z)\rangle = 0$.

Since T is normal, $\langle x - z, TT^*(y-z)\rangle = 0$. Thus we have $\langle T^*(x-z), T^*(y-z)\rangle = 0$.

Since T^* is linear, we have

$$\langle T^*(x) - T^*(z), T^*(y) - T^*(z)\rangle = 0.$$

Hence T^* is right-angle preserving. By Lemma 2.4, T and T^* are orthogonality preserving. By the second part of Theorem 2.3, we can prove that T is normal. ■

It is obvious that T^*T and $|T|$ are linear if $T \in B(H)$. So we obtain the following theorem.

2.7 Theorem. *Let H be a Hilbert space and $T \in B(H)$. If T is a right-angle preserving operator, then so are T^*T and $|T|$, where T^* is adjoint of T and $|T| = \sqrt{T^*T}$.*

2.8 Theorem. *Let T be a unitary operator in $B(H)$. Then T is orthogonality preserving and right-angle preserving.*

Proof. Suppose $x \perp y$, for all $x, y \in H$. Since every unitary operator on a Hilbert space H is isometry, we have

$$\langle T(x), T(y)\rangle = \langle x, y\rangle = 0.$$

Hence T is orthogonality preserving.

Again, suppose $x - z \perp y - z$, for all $x, y, z \in H$. Since every unitary operator on a Hilbert space H is isometry, we have

$$\langle T(x-z), T(y-z)\rangle = \langle x-z, y-z\rangle = 0.$$

Moreover T is linear, so $\langle T(x)-T(z), T(y)-T(z)\rangle = 0$. That is,

$$T(x)-T(z) \perp T(y)-T(z).$$

Hence T is right-angle preserving. ■

3 Stability Problems

In this section, we omit some properties of approximately orthogonality and approximate right-angle preserving operators. These operators can be extensively studied in $[2,3]$. We show the relation between approximate right-angle preserving and approximately orthogonality preserving. We present quasi-linear and some stability problems. We prove that Chmielinski's stability question for right-angle preserving and approximate right-angle preserving; see $[3]$.

3.1 Theorem. $[2]$ *Let X and Y be (real or complex) inner product spaces.*

Let $T: X \to Y$ be a nonzero linear operator satisfying approximately orthogonality preserving with some $\varepsilon \in [0,1)$. Then T is injective, continuous and there exists $\gamma > 0$ such that

(1) $$\left|\langle T(x), T(y)\rangle - \gamma^2 \langle x, y\rangle\right| \le \delta \min\left\{\gamma^2 \|x\| \ \|y\|, \|T(x)\| \ \|T(y)\|\right\},\ x,\ y \in X,$$

with $\delta = 4\varepsilon\left(\dfrac{1}{1-\varepsilon} + \sqrt{\dfrac{1+\varepsilon}{1-\varepsilon}}\right)$.

Conversely, if $T: X \to Y$ satisfies **(1)** *with some $\delta \ge 0$ and $\gamma > 0$, then T is a quasi-linear, approximately orthogonality preserving operator. More precisely, T satisfies*

$$\|T(x+y) - T(x) - T(y)\| \le 2\sqrt{\delta}\gamma(\|x\| + \|y\|),\ x,\ y \in X,$$

$$\|T(\lambda x) - \lambda T(x)\| \le 2\sqrt{\delta}\gamma |\lambda| \|x\|,\ x \in X,\ \lambda \in \mathbb{K}$$

and $x \perp y \Rightarrow T(x) \perp^{\delta} T(y); T(x) \perp T(y) \Rightarrow x \perp^{\delta} y$, *for* $x,\ y \in X$.

Proof. See $[2]$. ■

3.2 Theorem. $[3]$ *Let X and Y be real inner product spaces and $T: X \to Y$ be an operator. If T satisfies approximately orthogonality preserving and T is additive, then T satisfies approximate right-angle preserving and $T(0) = 0$.*

Proof. See $[3]$. ■

3.3 Theorem. *Let X and Y be real inner product spaces and $T: X \to Y$ be an operator. For $\varepsilon \in [0,1)$, if T is approximate right-angle preserving and linear, then T is approximately orthogonality preserving.*

Proof. If T is approximate right-angle preserving, then

$$x-z \perp y-z \text{ implies } T(x)-T(z) \perp^{\varepsilon} T(y)-T(z),$$

where $x,\ y,\ x-z,\ y-z \in X$.

Let $u = x-z,\ v = y-z.$ If $u \perp v$ then $\langle u,v \rangle = 0.$

$$\begin{aligned} \left|\langle T(u),T(v)\rangle\right| &= \left|\langle T(x-z),T(y-z)\rangle\right| \\ &= \left|\langle T(x)-T(z),T(y)-T(z)\rangle\right| \\ &\le \varepsilon \left\|T(x)-T(z)\right\| \ \left\|T(y)-T(z)\right\| \\ &= \varepsilon \left\|T(x-z)\right\| \ \left\|T(y-z)\right\| \\ &= \varepsilon \left\|T(u)\right\| \ \left\|T(v)\right\|. \end{aligned}$$

This means that $T(u) \perp^{\varepsilon} T(v).$ Hence every linear approximate right-angle preserving operator is approximately orthogonality preserving. ■

The stability of the orthogonality preserving property has been studied in $[4]$. The following theorem is used in Corollary 3.8.

3.4 Theorem. $[3]$ *Let X, Y be inner product spaces and let X be finite-dimensional. Then there exists a continuous function $\delta:[0,1) \to [0,+\infty)$ with the property $\lim_{\varepsilon \to 0^{+}} \delta(\varepsilon) = 0$ such that for each linear operator $f: X \to Y$ satisfying approximately orthogonality preserving one finds a linear, orthogonality preserving one $T: X \to Y$ such that*

$$\|f-T\| \le \delta(\varepsilon) \min\{\|f\|,\ \|T\|\}.$$

The mapping δ depends, actually, only on the dimension of X.

3.5 Lemma. $[2]$ *Let X and Y be two inner product spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose that $T: X \to Y$ satisfies, with some $\delta \ge 0$ and $\gamma > 0,$ the functional inequality*

(1) $$\left|\langle T(x),T(y)\rangle - \gamma^{2}\langle x,y\rangle\right| \le \delta\gamma^{2}\|x\|\ \|y\|,\ x,\ y \in X.$$

Then T is quasi-linear. That is, T is quasi-additive and quasi-homogeneous.

We need the following proposition for establishing our next corollary.

3.6 Proposition. $[3]$ *Let $f: X \to Y$ satisfy right-angle preserving and $f(0)=0.$ Suppose that $g: X \to Y$ satisfies*

$$\|f(x)-g(x)\| \le M\|f(x)\|,\ x \in X$$

with some constant $M < \frac{1}{4}$. Then g satisfies approximately orthogonality preserving *with* $\varepsilon = \dfrac{M(M+2)}{(1-M)^2}$ *and*

$$\|g(x+y)-g(x)-g(y)\| \le 2\sqrt{\varepsilon}\left(\|g(x)\|+\|g(y)\|\right),\ x,\ y \in X;$$

$$\|g(\lambda x)-\lambda g(x)\| \le 2\sqrt{\varepsilon}\,|\lambda|\,\|g(x)\|,\ x \in X,\ \lambda \in \mathbb{K}.$$

Proof. Suppose that $f : X \to Y$ *satisfy right-angle preserving and* $f(0)=0$. *By Theorem 2.5, f satisfies* $\|f(x)\| = \gamma\|x\|$ *and* $\langle f(x), f(y)\rangle = \gamma^2\langle x, y\rangle,\ x,\ y \in X$ for some constant $\gamma \ge 0$. And then f is linear. Thus we have for arbitrary $x,\ y \in X$:

$$\begin{aligned}
\left|\langle g(x), g(y)\rangle - \gamma^2\langle x, y\rangle\right| &= \left|\langle g(x), g(y)\rangle - \langle f(x), f(y)\rangle\right| \\
&= \left|\langle g(x), g(y)\rangle - \langle f(x), g(y)\rangle + \langle f(x), g(y)\rangle - \langle f(x), f(y)\rangle\right| \\
&= \left|\langle g(x)-f(x), g(y)-f(y)+f(y)\rangle + \langle f(x), g(y)-f(y)\rangle\right| \\
&= \left|\langle g(x)-f(x), g(y)-f(y)\rangle + \langle g(x)-f(x), f(y)\rangle + \langle f(x), g(y)-f(y)\rangle\right|
\end{aligned}$$

By Cauchy-Schwarz Inequality,

$$\begin{aligned}
\left|\langle g(x), g(y)\rangle - \gamma^2\langle x, y\rangle\right| &\le \|g(x)-f(x)\|\ \|g(y)-f(y)\| + \|g(x)-f(x)\|\ \|f(y)\| + \|f(x)\|\ \|g(y)-f(y)\| \\
&\le M^2\|f(x)\|\ \|f(y)\| + M\|f(x)\|\ \|f(y)\| + M\|f(x)\|\ \|f(y)\| \\
&= M^2\gamma^2\|x\|\ \|y\| + 2M\gamma^2\|x\|\ \|y\| \\
&= M(M+2)\gamma^2\|x\|\ \|y\|.
\end{aligned}$$

By Lemma 3.5, we get, for any $x,\ y \in X,\ \lambda \in \mathbb{K}$;

$$\begin{aligned}
\|g(x+y)-g(x)-g(y)\| &\le 2\sqrt{M(M+2)}\gamma\left(\|x\|+\|y\|\right) \\
&= 2\sqrt{M(M+2)}\left(\|f(x)\|+\|f(y)\|\right);
\end{aligned}$$

$$\begin{aligned}
\|g(\lambda x)-\lambda g(x)\| &\le 2\sqrt{M(M+2)}\gamma|\lambda|\|x\| \\
&= 2\sqrt{M(M+2)}|\lambda|\|f(x)\|.
\end{aligned}$$

Since $\|f(x)\| - \|g(x)\| \le M\|f(x)\|$,

$$\begin{aligned}
\|f(x)\| - M\|f(x)\| &\le \|g(x)\| \\
\|f(x)\| &\le \frac{\|g(x)\|}{1-M}.
\end{aligned}$$

Therefore

$$\left|\langle g(x), g(y)\rangle - \gamma^2 \langle x, y\rangle\right| \le M(M+2)\|f(x)\| \ \|f(y)\|$$
$$\le \frac{M(M+2)}{(1-M)^2}\|g(x)\| \ \|g(y)\|.$$

Letting $\varepsilon = \dfrac{M(M+2)}{(1-M)^2}$. According to converse of Theorem 3.1, we have $x \perp y$ implies $g(x) \perp^{\varepsilon} g(y)$ and

$$\|g(x+y) - g(x) - g(y)\| \le 2\sqrt{\varepsilon}\left(\|g(x)\| + \|g(y)\|\right), \ x, y \in X.$$
$$\|g(\lambda x) - \lambda g(x)\| \le 2\sqrt{\varepsilon}|\lambda|\|g(x)\|, \ x \in X, \ \lambda \in \Box.$$

■

3.7 Corollary. $[3]$ *If* f *satisfies right-angle preserving,* $f(0)=0$ *(whence* f *is linear) and* $g: X \to Y$ *is a linear operator satisfying, with* $M < \frac{1}{4}$*,*

$$\|f - g\| \le M\|f\|,$$

then g *is approximate right-angle preserving.*

Proof. According to the above Proposition 3.6, we get g is approximately orthogonality preserving and since g is linear. It follows from Theorem 3.2 that g is approximate right-angle preserving. ■

According to the reverse statement in Corollary 3.7, we show the following corollary.

3.8 Corollary. *If a linear operator* $g: X \to Y$ *satisfies approximate right-angle preserving (with some* $\varepsilon > 0$ *), then there exists a linear operator* f *satisfying right angle preserving such that an estimation of*

$$\|f - g\| \le M\|f\|, \text{ with } M < \frac{1}{4}.$$

Proof. By Theorem 3.3, if g is approximate right-angle preserving and linear, then g is approximately orthogonality preserving. It follows from Theorem 3.4 that there exists linear f and satisfying orthogonality preserving, hence f is a right-angle preserving operator such that

$$\|g - f\| \le \delta(\varepsilon)\min\{\|g\|, \ \|f\|\}$$
$$\|f - g\| \le M\|f\|.$$

■

Conclusion

This research paper discusses orthogonality preserving and right-angle preserving operators on inner product spaces and Hilbert spaces. We see that some properties of bounded linear orthogonality preserving operators and stability of orthogonality property are also true for right-angle preserving. We finally present the stability of two types of preserving operators.

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